Example 2. The joint density function for a pair of random variables X and Y

$$f(x, y) = \begin{cases} 0.1e^{-0.5x-0.2y} & \text{if } x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\rightarrow e^{-0.5x}\, e^{-0.2y}$$

   a. Verify that f is a joint density function.
   b. Find the expected values of X and Y.

Hint: $\int u e^{-u}\, du = -u e^{-u} - e^{-u}$

a. We check if $\iint_{R^2} f(x,y)\, dA = 1$:

$$\iint_{R^2} f(x,y)\, dA = \int_0^\infty \int_0^\infty 0.1 e^{-0.5x} e^{-0.2y}\, dy\, dx = \int_0^\infty \left[-\tfrac{1}{2} e^{-0.5x} e^{-0.2y} \right]_{y=0}^\infty dx$$

$$= \int_0^\infty \tfrac{1}{2} e^{-0.5x}\, dx = \left[-e^{-0.5x} \right]_{x=0}^\infty = 1 \quad \checkmark$$

We also check if $f(x,y) \geq 0$ for all x, y:

$$0.1 e^{-0.5x - 0.2y} \geq 0 \quad \text{no matter what } x \text{ and } y \text{ are.} \quad \checkmark$$

b. expected value of $X = \int_0^\infty \int_0^\infty 0.1 x\, e^{-0.5x} e^{-0.2y}\, dy\, dx$

$$= \int_0^\infty \left[-\tfrac{1}{2} x e^{-0.5x} e^{-0.2y} \right]_{y=0}^\infty dx = \int_0^\infty \tfrac{1}{2} x e^{-\frac{1}{2}x}\, dx = 2 \int_0^\infty u e^{-u}\, du$$

Let $u = \tfrac{1}{2} x$
$du = \tfrac{1}{2} dx$

$$= 2 \left[-u e^{-u} - e^{-u} \right]_{u=0}^\infty = 2(1) = 2$$

c. expected value of $Y = \int_0^\infty \int_0^\infty 0.1 y\, e^{-0.5x} e^{-0.2y}\, dx\, dy$

$$= \int_0^\infty \left[-\tfrac{1}{5} y e^{-0.5x} e^{-0.2y} \right]_{x=0}^\infty dy = \int_0^\infty \tfrac{1}{5} y e^{-\frac{1}{5}y}\, dy = 5 \int_0^\infty u e^{-u}\, du$$

Let $u = \tfrac{1}{5} y$
$du = \tfrac{1}{5} dy$

$$= 5 \left[-u e^{-u} - e^{-u} \right]_{u=0}^\infty = 5(1) = 5$$